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Letter

Diffraction efficiency of volume gratings with finite size: corrected analytical solution

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Abstract. In order to correct errors in a previously published work on a treatment of two-dimensional coupled-wave theory, a complete derivation of the two-dimensional 'overlap grating' coupled wave equations is given. By using the Riemann method, a corrected solution to the equations in closed mathematical form is obtained. On the basis of this solution a brief investigation of the diffraction properties of finite-sized gratings, and in particular the dependence of diffraction efficiency on the geometric size of gratings, is given.

As diffractive optical elements, volume gratings have been widely used in fields such as information storage, processing, and display, and for optoelectronic devices such as waveguides and filters [1–3]. The diffraction properties of volume gratings were presented in an elegant form by Kogelnik [4], who used coupled-wave theory. Due to the one-dimensional nature of his theory, it is oriented, in principle, to transmission or reflection gratings where the area of the entrance and exit surfaces (corresponding to the transverse apertures of the incident and diffracted beams, respectively) are much larger than the grating thickness. However, in many contemporary applications the size of the volume grating in the two dimensional (2-D) plane in which the grating vector lies is restricted. In addition, the two beams involved in grating recording or reconstruction may access two orthogonal surfaces (typically called the 90° geometry [5]). Analysis of the diffraction properties of these 2-D restricted volume gratings requires more accurate treatment, hence two-dimensional grating theory has long been a subject of considerable attention [6–9].

A two-dimensional grating theory assumes that there is no change in the direction perpendicular to the grating plane ($x-y$ plane), which is usually the chosen direction of the electric field vector. Based on the 2-D coupled-wave theory developed by Solymar and coworkers [6–8], Russell and Solymar investigated the diffraction properties of 'overlap gratings' [9]. In such an overlap grating, the

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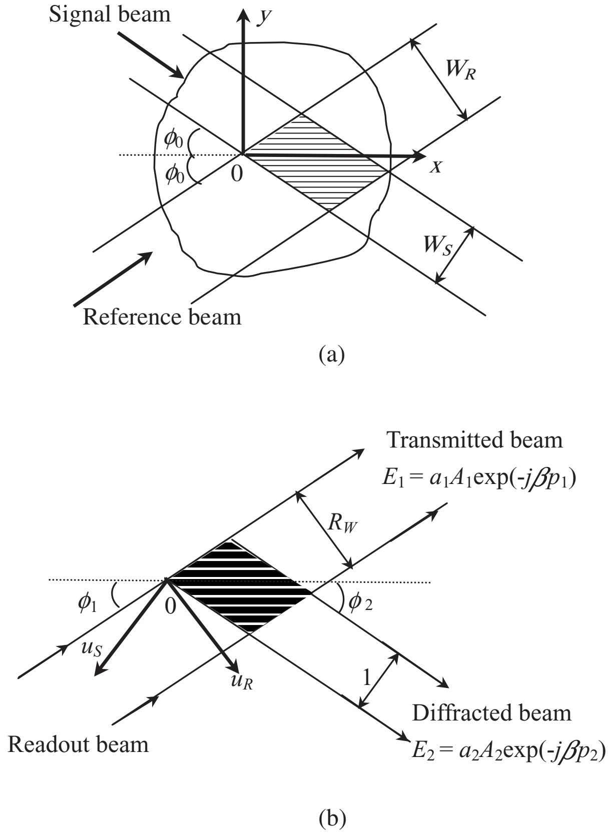


Figure 1. Formation of an overlap volume grating (a) and Readout of the developed grating (b).

grating is formed only in the area where two plane waves of finite widths intersect in the interior of a larger recording material, as shown in figure 1(a). Russell and Solymar presented a solution to the 2-D differential equations in closed mathematical form, which should have been able to deal with rather general cases of overlap gratings [9]. For example, a simplified form of their solution (for lossless gratings and Bragg readout) was used to design holographic finite-beam converters with perfect fidelity [10]. However, as pointed out by several authors (e.g. see references [11, 12]), there are some critical typographical errors in their solution, and in particular in 'equation (11)' from reference [9], which we show as equation (A1) in the Appendix of this Letter. For instance, diffraction efficiencies calculated with this equation do not scale with grating strength or absorption in a logical way.

The aim of the present letter is to correct these errors from reference [9], and to provide a corrected, generally-applicable solution to the 2-D coupled-wave equations. On the basis of the revised mathematical solution, a brief investigation of the diffraction efficiency of 2-D restricted gratings is given.

For convenience and in comparison with reference [9], the notations used in this Letter are identical to those in reference [9]. We start first with a grating written by two plane waves, as shown in figure 1(a), each with propagation constant $\beta_0 = 2\pi/\lambda_0$. Thus, the grating can be specified by the modulated dielectric constant of the medium, ε_r, written as

$$\varepsilon_r = \varepsilon_{r0} + \varepsilon_{r1} a_{10} a_{20} \cos[\beta_0(p_{10} - p_{20})], \tag{1}$$

where a_{10} and a_{20} are the normalized amplitude distributions across the plane wavefronts of the reference and object beam respectively, and p_{i0} is the plane phase factor of the beams, written as $p_{i0} = x\cos\phi_0 - (-1)^i y\sin\phi_0$. Here, ε_{r0} is the average dielectric constant, and ε_{r1} is the modulation depth, which may be real (complex) to indicate a pure phase (phase and absorption) grating.

When the grating is read out in the Bragg regime, there are only two beams present in the grating, as shown in figure 1 (b): the readout beam $E_1 = a_1 A_1 \exp(-j\beta p_1)$ and diffracted signal $E_2 = a_2 A_2 \exp(-j\beta p_2)$. The purpose of having two amplitude coefficients is that the a_i terms describe the normalized transverse amplitude distribution across the beam, while the A_i terms indicate the change in amplitude along the propagation direction. The total electrical field E is the sum of E_1 and E_2:

$$E = E_1 + E_2 = a_1 A_1 \exp(-j\beta p_1) + a_2 A_2 \exp(-j\beta p_2), \tag{2}$$

where $\beta = 2\pi/\lambda$ is the propagation constant of the readout beam (allowing changes in readout wavelength from λ_0 to λ), p_1 and p_2 the phase fronts of the two beams (dictated by angle ϕ rather than ϕ_0). Here, it is assumed that the readout beam completely covers the grating region, even when ϕ differs from ϕ_0. Note that the expression of E_i is essentially the same as what was implicitly specified in [9], with the field amplitudes, A_1 and A_2, varying with spatial coordinates due to both material absorption as well as wave coupling between each other through the dielectric grating.

Wave propagation in a lossy material is described by the scalar wave equation:

$$\nabla^2 E + \left(\frac{\omega^2}{c^2}\varepsilon_r - j\omega\sigma\mu\right)E = 0. \tag{3}$$

Combining equations (1–3), and using similar approximations to those used in reference [9], lead to the coupled wave equations expressed as follows:

$$\nabla A_1 \cdot \nabla p_1 = -\alpha A_1 - j\kappa \frac{a_{20} a_{10} a_2}{a_1} \exp(-jK) A_2, \tag{4}$$

$$\nabla A_2 \cdot \nabla p_2 = -\alpha A_2 - j\kappa \frac{a_{20} a_{10} a_1}{a_2} \exp(jK) A_1, \tag{5}$$

where $\kappa = (\varepsilon_{r1}\beta_0)/(4\varepsilon_r)$ is the coupling strength, $\alpha = (\omega\sigma\mu)/2\beta$ is the material absorption coefficient of the field amplitude, and $K = \beta_0(p_{10} - p_{20}) - \beta(p_1 - p_2)$ is the Bragg mismatch parameter. Note that although our expressions for E_i are the same as those in reference [9], the coupled-wave equations (4–5) are different from equation (7) in reference [9] [shown as equation (A2) in the Appendix of this Letter]. However, if we separate the absorption from A_i by $A_i = A_i^{(R)} \exp(-\alpha p_i)$, the differential equations that $A_i^{(R)}$ satisfies would become identical to equation (A2).

It is convenient to follow the evolution of A_i along their respective propagation directions, which can be implemented by using the coordinate transformation from reference [9],

$$\begin{pmatrix} u_R \\ u_S \end{pmatrix} = \frac{1}{W_s} \begin{pmatrix} \sin\phi_0 & -\cos\phi_0 \\ -\sin\phi_0 & -\cos\phi_0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{6}$$

The amplitude functions a_1 and a_2 can then be represented by $a_1(u_R)$ and $a_2(u_S)$ respectively, where the grating is defined in the range $-1 < u_S < 0$ and $0 < u_R < R_W$, and $R_W = W_R/W_S$ is the width ratio of reference to signal beam.

Equations (4) and (5) can be rewritten

$$\frac{\partial A_1}{\partial u_S} = \alpha' W_S A_1 + j\kappa' \frac{a_{20} a_{10} a_2}{a_1} \exp(-jK) W_S A_2, \tag{7}$$

$$\frac{\partial A_2}{\partial u_R} = -\alpha' W_S A_2 - j\kappa' \frac{a_{20} a_{10} a_1}{a_2} \exp(jK) W_S A_1, \tag{8}$$

where $\alpha' = \alpha/\sin 2\phi_0$ and $\kappa' = \kappa/\sin 2\phi_0$. In deriving equations (7–8), we considered the Bragg regime only, so that the angles ϕ_1 and ϕ_2, appearing in the phase functions p_1, and p_2, were assumed to be equal to ϕ_0 except in the Bragg mismatching parameter K, which is

$$K = \delta W_s(u_R + u_S), \quad \delta = \frac{1}{2}\beta_0 \sec^2\phi_0 \Delta\phi + \beta_0 \frac{\Delta\lambda}{\lambda_0} \tan\phi_0. \tag{9}$$

Here $\Delta\phi = \phi_1 - \phi_0$ and $\Delta\lambda = \lambda_1 - \lambda_0$ are the angle and wavelength deviations from the Bragg condition, respectively. Note that equation (9) differs from the expression for δ in reference [9].

Combining (7–9) and letting

$$A = A_2 \exp[-(\alpha' W_S + j\delta W_S) u_S] \exp(\alpha' W_S u_R), \tag{10}$$

and with the assumption $a_2 = a_{20}$, we obtain the differential equation for function A:

$$\frac{\partial^2 A}{\partial u_R \partial u_S} - \kappa'^2 W_S^2 a_{20}^2 a_{10}^2 A = 0. \tag{11}$$

Without the assumption that $a_2 = a_{20}$, equation (11) would be too complicated to solve. But since a_2 is essentially an 'initial distribution' and the initial value of E_2 is zero, a_2 can be arbitrarily assigned. Thus the assumption that $a_2 = a_{20}$ simply implies that any subsequent transverse distribution in E_2 that differs from a_{20} is attributed to the transverse distribution of A, which becomes a 2-D variable.
The boundary conditions of equation (11) are as follows:

i. On $u_R = 0, A_2(0, u_S) = 0$, and $\left.\frac{\partial A_2}{\partial u_S}\right|_{u_R=0} = 0$, so $A(0, u_S) = 0$, and $\left.\frac{\partial \mathrm{A}}{\partial u_S}\right|_{u_R=0} = 0$; (12)

ii. On $u_S = 0$, $A_1 = A_1$ (u_R, 0), which may be an arbitrarily given function. Even for the simplest case where $A_1 = 1$ across the readout beam, if the surrounding medium has the same absorption coefficient as the grating area, then the amplitude distribution across the input boundary ($u_S = 0$) must be non-uniform if the angle

ϕ_0 differs from 45°, i.e.

$$A_1(u_R, 0) = \exp(-\alpha p_1) = \exp(-\alpha' W_S \cos 2\phi_0 u_R). \tag{13}$$

Thus, A (u_R, 0) can be solved from equations (10) and (8), and its partial derivative with respect to u_R can be expressed as

$$\left.\frac{\partial A}{\partial u_R}\right|_{u_S=0} = -j\kappa' W_S a_{10} a_1 \exp\left[\left(2\alpha' \sin^2\phi_0 + j\delta\right) W_S u_R\right]. \tag{14}$$

The solution of the standard hyperbolic differential equation (11), subject to the boundary conditions equations (12–14), can be obtained by Riemann method [13], leading to an expression in closed mathematical form:

$$A(u_R, u_S) = -j\kappa' W_S \int_0^{u_R} a_{10}(\tau) a_1(\tau) J_0\left(2\kappa' W_S \sqrt{LM}\right) \exp\left[\left(2\alpha' \sin^2\phi_0 + j\delta\right) W_S \tau\right] d\tau, \tag{15}$$

where

$$L = \int_\tau^{u_R} a_{10}^2(\xi)\, d\xi, \quad M = \int_{u_S}^0 a_{20}^2(\xi)\, d\xi. \tag{16}$$

Thus A_2 can be solved from equation (10), and A_1 can be found by differentiating A_2 and using equations (8) and (10). Omitting the phase factor $\exp(-j\beta p_i)$, the final solution is

$$\begin{aligned} E_1 = {} & \exp[-\alpha' W_S (u_R \cos 2\phi_0 - u_S)] a_1(u_R) - \exp[-\alpha' W_S (u_R - u_S)] \exp(-j\delta W_S u_R) \\ & \times a_{10}(u_R) \kappa' W_S \int_0^{u_R} a_{10}(\tau) a_1(\tau) \sqrt{\frac{M}{L}} J_1\left(2\kappa' W_S \sqrt{LM}\right) \exp\left[\left(2\alpha' \sin^2\phi_0 + j\delta\right) W_S \tau\right] d\tau, \end{aligned} \tag{17}$$

for the amplitude of the transmitted field, and

$$\begin{aligned} E_2 = {} & -j\kappa' W_S a_2(u_S) \exp[-\alpha' W_S (u_R - u_S)] \exp(j\delta W_S u_S) \\ & \times \int_0^{u_R} a_{10}(\tau) a_1(\tau) J_0\left(2\kappa' W_S \sqrt{LM}\right) \exp\left[\left(2\alpha' \sin^2\phi_0 + j\delta\right) W_S \tau\right] d\tau, \end{aligned} \tag{18}$$

for the diffracted field.

The diffraction efficiency can be defined as the ratio of diffracted to incident power,

$$\eta_{diff} = \frac{\displaystyle\int_{-1}^0 \left|E_2(R_W, u_S)\right|^2 du_S}{\displaystyle\int_0^{R_W} \left|E_1(u_R, 0)\right|^2 du_R}. \tag{19}$$

In the simplest case of plane wave gratings ($a_{i0} = a_i = 1$) with $\alpha = \delta = 0$, we obtain the same analytical expression for the diffraction efficiency,

$$\eta_{diff} = 1 - J_0^2\left(2\kappa' W_S \sqrt{R_W}\right) - J_1^2\left(2\kappa' W_S \sqrt{R_W}\right), \tag{20}$$

that was originally derived in reference [7] and discussed in reference [9]. Figure 2 shows the diffraction efficiency against the equivalent grating strength $\kappa' W_S \sqrt{R_W}$ calculated using equation (20). It can be seen that for lossless gratings the

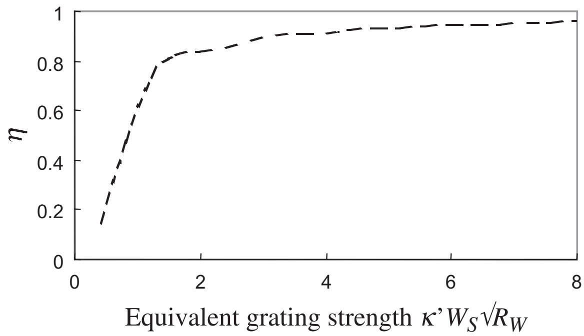


Figure 2. Bragg-matched diffraction efficiency against the equivalent grating strength, $\alpha = 0$.

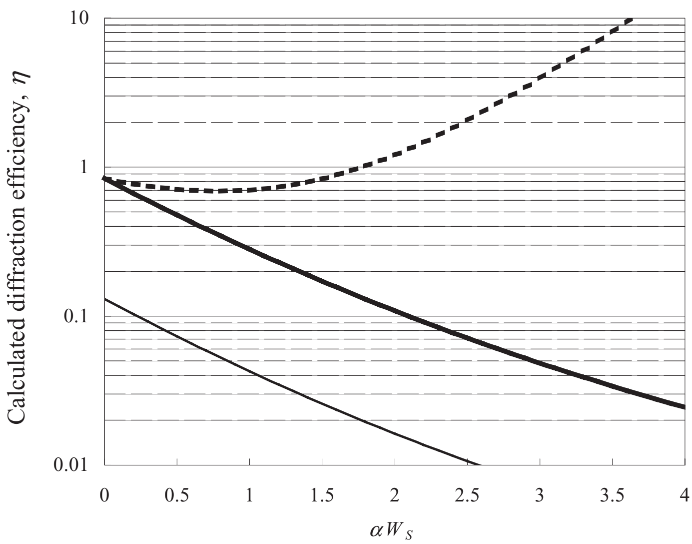


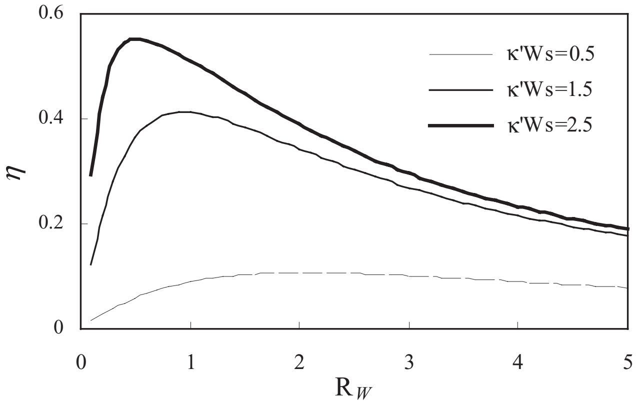
Figure 3. Calculated diffraction efficiency using uncorrected equation (A1) (solid thin line), equation (A1) with the corrected upper-limit to the integration (dashed black line), and our corrected solution equation (18) (solid black line), respectively ($\phi_0 = 45^\circ$, $\kappa' W_S = 2$, $\delta = 0$, $R_W = 1$).

efficiency increases with increasing width of the reference beam if the coupling strength $\kappa' W_S$ is fixed.

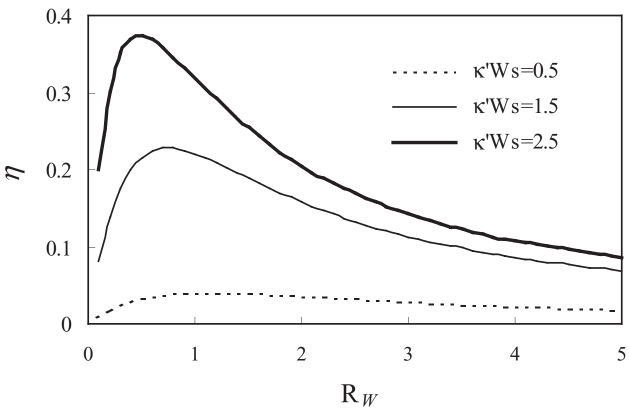
When α and/or δ is non-zero, equation (20) is invalid, but one can calculate diffraction efficiencies by using equations (17–19).

Comparing our solutions (17) and (18) with the previous solution (A1), one can see that, apart from some insignificant problems such as a slightly different expression for the exponential function under integration, there are two main typographical errors in (A1): (i) an incorrect upper-limit to the integration (u_S instead of u_R) that caused the earlier calculation result to scale incorrectly with grating strength; and (ii) an incorrect sign of u_S in the attenuation term outside the integration that made the calculated diffraction efficiency much larger than unity for some values of $\alpha > 0$, even if (i) were to be corrected. Figure 3 shows typical curves calculated using the uncorrected or partially corrected solution, and a curve using equations (17–19) for comparison.

Equations (17–19) can be used for any complicated overlap gratings including non-uniform writing beams ($a_i \neq$ constant), absorption materials ($\alpha > 0$), phase and absorption gratings (complex κ), arbitrary writing angles ($\phi_0 \neq 45°$) and non-Bragg readout ($\delta \neq 0$). As an example, the effect of the geometrical size on the diffraction efficiency of lossy gratings is studied using equations (17–19). Figure 4 shows the dependence of efficiency on grating size ratio R_W for different $\kappa' W_S$ and $\alpha' W_S$. Comparing with figure 2, one can see that the relationship between η and R_W is no longer monotonic if $\alpha > 0$. It is obvious that in order to optimize the diffraction efficiency of lossy gratings, the geometrical dimensions of the grating should be carefully designed. For truncated-overlap or non-uniform gratings (κ is non-constant), a numerical method may be more applicable.



(a)



(b)

Figure 4. Diffraction efficiency as function of grating size ratio R_W for different $\kappa' W_S$, (a) $\alpha' W_S = 0.5$, (b) $\alpha' W_S = 1$. ($\phi_0 = 45°$).

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Appendix

The solution obtained in reference [9] is

$$\left.\begin{aligned} E_1 &= \exp(\alpha' W_S u_S) \times \Big\{a_1(u_R) - \kappa' W_S \exp[-(\alpha' W_S + j\delta W_S)u_R] a_{10}(u_R) \\ &\quad \times \int_0^{u_S} a_1(v)\sqrt{\frac{L}{M}} J_1\Big(2\kappa' W_S\sqrt{LM}\Big) \exp[v(\alpha' W_S + j\delta W_S)]\, dv\Big\}, \\ E_2 &= j\kappa' W_S a_{20}(u_S) \exp[-\alpha' W_S(u_R + u_S) - j\delta W_S u_S] \\ &\quad \times \int_0^{u_S} a_1(v) J_0\Big(2\kappa' W_S\sqrt{LM}\Big) \exp[v(\alpha' W_S + j\delta W_S)]\, dv, \end{aligned}\right\} \tag{A1}$$

which is the solution to the coupled wave equations

$$\left.\begin{aligned} \nabla A_1 \cdot \nabla p_1 + j\kappa \frac{a_{20}a_2a_{10}}{a_1} \exp[-(G + jK)]A_2 = 0, \\ \nabla A_2 \cdot \nabla p_2 + j\kappa \frac{a_{20}a_{10}a_1}{a_2} \exp[(G + jK)]A_1 = 0, \end{aligned}\right\} \tag{A2}$$

where

$$G = \alpha(p_2 - p_1),\ K = \beta(p_{10} - p_{20}) - \beta(p_1 - p_2),\ \kappa = \varepsilon_{r1}\beta_0/(4\varepsilon_r). \tag{A3}$$

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